B.A./B.Sc. Part III (Honours) Examination, 2021 (1+1+1)
Subject: Mathematics
Paper V

Time:2 Hours Full Marks: 50

The figures in the margin indicate full marks.
Candidates are required to write their answers in their own words as far as practicable.
[Notation and Symbols have their usual meaning]

1. Answer any four questions: $4\times5 = 20$

(a) State and prove monotone convergence theorem for a sequence. Also prove that a monotone increasing sequence which is unbounded above diverges to infinity [3+2]

(b) State Cauchy's convergence criterion for series and using it, prove that the necessary condition for the convergence of a series $\sum_{n=1}^{\infty} a_n$ is that the n th term a_n must tend to zero as n tends to infinity. Does the converse of this result hold? Justify your answer. [2+2+1]

(c) Expand Fourier series for $f(x) = x^2$ in $(0, 2\pi]$. [5]

(d) Show that in a complete metric space (X, d), a subspace (Y, d_Y) of (X, d) is complete if and only if it is closed. [5]

(e) Prove that in a metric space arbitrary union of open sets is open. Does this result hold for arbitrary intersections? Justify your answer. Prove that every metric space is first countable. [2+1+2]

(f) Let $f: G \to \mathbb{C}$, where $f(x+iy) = u(x,y) + iv(x,y)$ be a function of complex variable $z = x+iy$ in a region G. Let u, v be differentiable at (x_0, y_0) and satisfy Cauchy Riemann equations at (x_0, y_0). Show that f is differentiable at $z_0 = x_0 + iy_0$. [5]

2. Answer any three questions: $3\times10 = 30$

(a) (i) Examine for uniform convergence of the sequence of functions $f_n(x) = \frac{nx^2}{1+nx}$ on $[0, 1]$. [3]

(ii) If a sequence of continuous functions $\{f_n\}$ converges uniformly to a function f on $[a, b]$ then show that f is also continuous on $[a, b]$. [4]

(iii) Examine the uniform convergence of the series of functions $\sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)}$ on $[0, 2]$. [3]

(b) (i) Show that $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$ [5]

(ii) Find the extreme values of the function $f(x,y,z) = x^2 + y^2 + z^2$ subject to the condition $x^2 - z^2 = 1$ [5]

(c) (i) Show that compact metric space is sequentially compact. [3]

(ii) Prove that a function $f:(X,d) \to (Y,\rho)$ is continuous if and only if inverse image $f^{-1}(F)$ is a closed subset of X for every closed subset F of Y. [4]

(iii) Prove that totally bounded metric space is separable. [3]

(d) (i) Find radius of convergence of the power series $\sum_{n=2}^{\infty}(logn)^2 z^n$. [3]

(ii) Find the bilinear transformation which transforms $z_1 = 2, z_2 = 1, z_3 = 0$ into $w_1 = 1, w_2 = 0, w_3 = i$. [4]

iii) Let f be an analytic function in a region G. If f is real valued then show that f is constant in G. [3]

(e) (i) Examine the convergence of the following: [3+3]

(a) $\sum_{n=2}^{\infty}\frac{1}{n(logn)^2}$

(b) $1+\left(\frac{2}{3}\right)^2+\left(\frac{2.4}{3.5}\right)^2+\left(\frac{2.4.6}{3.5.7}\right)^2+\ \dots$

(ii) Show that $(C[a,b],d)$ forms a complete metric space, where $C[a,b]$ is the space of all continuous functions on $[a,b]$ and the metric d is given by [4]

$$d(f,g) = \operatorname*{Sup}_{t\in[a,b]} |f(t) - g(t)|,\ f,g \in C[a,b].$$